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SYMMETRIZATION OF THE SIGN-DEFINITENESS CRITERIA OF SYMMETRICAL QUADRATIC FORMS†

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The symmetry of an expression means that, among the parameters and variables, there are certain groups such that the form of the expression does not change for the one-and-the same permutation of the elements (or for the permutation of these indices) in each of the groups. It is proved that, in the case of symmetrical quadratic forms, symmetrization of the Silvester criterion and the Mann criterion by summation of the left-hand sides of inequalities, using the above-mentioned permutation of the indices, gives the sign-definiteness criteria. The proof is carried out by induction with respect to the number of variables occurring in the quadratic form under the assumption that, in the vector of the variables, the elements of each group do not alternate with other variables. The problem of the stability of the orientations of a satellite-gyrostat in a circular orbit with a subsatellite tethered to it is considered as an example. © 2003 Elsevier Science Ltd. All rights reserved.

The use of the criteria for the positive-definiteness of quadratic forms in mechanics is associated above all with the stability analysis [1]. In problems of rigid body dynamics, the moments of inertia (A_1, A_2, A_3), the components of the angular velocity (ω_1, ω_2, ω_3) and the direction cosines (α_1, α_2, α_3), (β_1, β_2, β_3), (γ_1, γ_2, γ_3), for example, may be groups in which the expressions for the fundamental dynamic quantities do not change under a simultaneous cyclic permutation of the parameters or the variables. The existence of symmetry facilitates analytic calculations and makes them clearer and more compact.

In order to conserve symmetry, it is sometimes advisable not to eliminate redundant variables, and not to use the Silvester criterion [1] for the sign-definiteness of a quadratic form and the Mann criterion [2] for the sign-definiteness of a quadratic form in a linear manifold, which lead to loss of this symmetry.

1. SYMMETRIZATION OF THE SILVESTER CRITERION

In problems of mechanics, the sizes of the groups defining the symmetry of an expression is usually equal to 3 or 2, and a cyclic permutation of the indices is considered. To be specific, we shall carry out the subsequent discussion for a typical case when $s = 3$.

Consider the quadratic form

$$\varphi(x) = x^T A x, \quad A^T = A$$

Suppose that, among the variables $x_1, \dots, x_n$, groups (β_1, β_2, β_3), ..., (γ_1, γ_2, γ_3) are picked out

$$x^T = (x_1, \dots, x_m, (\beta_1, \beta_2, \beta_3), \dots, (\gamma_1, \gamma_2, \gamma_3))$$

and, among the parameters, on which the elements of the matrix A depend, groups (c_1, c_2, c_3), ..., (d_1, d_2, d_3) are picked out

$$A = A(b_1, \dots, b_l, (c_1, c_2, c_3), \dots, (d_1, d_2, d_3))$$

in which we shall carry out a cyclic permutation of the indices (123) without extending it to the variables $x_1, \dots, x_m$ and the parameters $b_1, \dots, b_l$.

A quadratic form $\varphi(x)$ is assumed to be symmetrical in the sense that its-form does not change on permuting the indices (123) in the groups which have been separated out. This means that a further two terms, obtained frow one another by cyclic permutation of the indices (123), correspond to each asymmetrical term in the quadratic form φ. On cyclic permutation of the indices, they simply change positions.

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We shall denote the operation of cyclic permutation by a prime

$$\begin{aligned}&(123)'=(231),\quad (231)'=(312),\quad (312)'=(123)\\&x'^T=(x_1,\ldots,x_m,(\beta_2,\beta_3,\beta_1),\ \ldots,\ (\gamma_2,\gamma_3,\gamma_1))\\&A'=A(b_1,\ldots,b_l,(c_2,c_3,c_1),\ \ldots,\ (d_2,d_3,d_1))\\&x'^TA'x'=x^TAx\end{aligned}$$

We shall denote the operation of summation with respect to a cyclic permutation of the indices (123) in the groups of variables and parameters which have been picked out by Σ. For example

$$\begin{aligned}&\Sigma(c_2+c_3)\beta_1^2=(c_2+c_3)\beta_1^2+(c_3+c_1)\beta_2^2+(c_1+c_2)\beta_3^2=\\&=\Sigma(c_3+c_1)\beta_2^2=\Sigma(c_1+c_2)\beta_3^2=\Sigma c_1(\beta_2^2+\beta_3^2)\end{aligned}$$

The principal diagonal minors Δ_i of order i of the matrix A are on the left-hand sides of the inequalities of the Silvester criterion for the positive-definiteness of a quadratic form $\varphi(x)$. The minor Δ_i can be considered as the determinant of the matrix $A^{(i)}$ of the quadratic form

$$\varphi^{(i)}(x^{(i)})=\frac{1}{2}x^{(i)T}A^{(i)}x^{(i)}=\varphi(x)\Big|_{x_{i+1}=\ldots=x_n=0},\quad x^{(i)}=\mathrm{col}(x_1,\ldots,x_i) \tag{1.1}$$

which is obtained from $\varphi(x)$ by equating the last $n-i$ components of the vector x to zero. If Δ_i corresponds to the part of the vector x with a cut-down cyclic group of variables, then the magnitude of Δ_i will change on cyclic permutation of the indices. Hence, we obtain three versions of the Silvester criterion

$$\Delta_1>0,\ \ldots,\ \Delta_n>0;\quad \Delta_1'>0,\ \ldots,\ \Delta_n'>0;\quad \Delta_1''>0;\ \ldots,\ \Delta_n''>0 \tag{1.2}$$

The basic result of this section lies in the proof that the symmetrized conditions

$$\Sigma\Delta_1>0,\ \ldots,\ \Sigma\Delta_n>0 \tag{1.3}$$

also give the criterion for the positive-definiteness of the quadratic form φ. Here, we have in mind a summation with respect to a cyclic permutation of the indices of the parameters of the separated groups which implicitly occur in the determinants Δ_i. These parameters appear explicitly in the expanded expressions for the determinants Δ_i.

The necessity of conditions (1.3) is obvious, since the sum of positive numbers is always positive. The sufficiency is not so trivial. We will now show that not every summation of the left-hand sides of the Silvester criterion with respect to permutation of the components of the vector x gives a criterion of positive-definiteness. For example, in the case of an incomplete summation (two terms) with respect to the cyclic permutation of the variables $(x_1, x_2, x_3)' = (x_2, x_3, x_1)$ and the parameters $(c_1, c_2, c_3)' = (c_2, c_3, c_1)$ for an alternating-sign quadratic form

$$\Sigma c_1x_1^2=2x_1^2-x_2^2-3x_3^2=-x_1'^2-3x_2'^2+2x_3'^2$$

we obtain

$$\Delta_1+\Delta_1'=2-1>0,\quad \Delta_2+\Delta_2'=-2+3>0,\quad \Delta_3+\Delta_3'=6+6>0$$

Before proving the sufficiency of the symmetrized Silvester criterion (1.3), we will present a further well-known criterion of positive-definiteness. It is obvious that, whatever the quadratic form $\varphi(x)$, a large number $\sigma^*>0$ is found such that the quadratic form $\Phi(\sigma)=\varphi(x)+\sigma x^2$ will be positive definite when $\sigma\geqslant\sigma^*$. It also remains positive definite when $\sigma=0$ if, as σ varies continuously from σ^* to zero, the determinant of the matrix of the quadratic form $\Phi(\sigma)$ (henceforth E_i is an $(i\times i)$ identity matrix)

$$\det(A+\sigma E_n)=\sigma^n+a_1\sigma^{n-1}+\ldots+a_n \tag{1.4}$$

does not degenerate or, what is the same thing, does not have negative roots, which will occur in this and only in this case when

$$a_1>0,\ \ldots,\ a_n>0 \tag{1.5}$$

The coefficients a_i are symmetrical functions (in the above-mentioned sense) since they are equal to the sums C_n^i of the diagonal minors of order i of the matrix A and these sums, in particular, also contain a summation over a cyclic permutation of the indices (123) in the groups which have been separated out. Hence conditions (1.5) also give a symmetrical criterion for the positive-definiteness of the quadratic form $\varphi(x) = \Phi(0)$, but they have a more complex structure than criteria (1.2) and (1.3).

We shall prove the sufficiency of criterion (1.3) by induction with respect to the number of variables occurring in the quadratic form $\varphi^{(i)}(x^{(i)})$ (1.1), under the assumption that the variables of each group do not alternate with the other variables in the vector of the variables x. We will carry out the induction, while conserving the symmetry of the function $\varphi^{(i)}(x^{(i)})$.

We take $\varphi^{(1)}(x_1)$ as the basis of the induction if x_1 does not occur in any cyclic group, and $\varphi^{(3)}(x^{(3)})$ if x_1, x_2, x_3 form a cyclic group. Neither quadratic forms change under a cyclic permutation of the indices and the criterion (1.3) holds, since it is identical to condition (1.5) for both of these quadratic forms.

Thus, suppose criterion (1.3) holds for a symmetrical quadratic form $\varphi^{(i)}$. We shall prove that it holds for $\varphi^{(i+1)}(x^{(i+1)})$ if x_{i+1} does not occur in the cyclic groups and, for $\varphi^{(i+3)}(x^{(i+3)})$, if $x_{i+1}, x_{i+2}, x_{i+3}$ form a cyclic group.

We will first consider the case of the quadratic form $\varphi^{(i+1)}(x^{(i+1)})$. We will prove that it is positive definite when a condition of type (1.3) is satisfied

$$\Sigma\Delta_1 > 0, \ldots, \Sigma\Delta_{i+1} > 0 \tag{1.6}$$

It follows from the first i conditions of (1.6), under the assumption of the induction, that the quadratic form $\varphi^{(i)}(x^{(i)})$ is positive definite and the Silvester conditions (1.2) are therefore satisfied by it

$$\Delta_1 > 0, \ldots, \Delta_i > 0 \tag{1.7}$$

By virtue of the assumption of the symmetry of the quadratic form $\varphi^{(i+1)}(x^{(i+1)})$, the largest principal diagonal minor Δ_{i+1} is independent of a cyclic permutation of the indices and is identical to the left-hand side of the last inequality in (1.6), apart from a factor of 3. This inequality and conditions (1.7) constitute the Silvester conditions for $\varphi^{(i+1)}(x^{(i+1)})$, and also proves that it is positive definite.

We will now prove the positive-definiteness of the quadratic form $\varphi^{(i+3)}(x^{(i+3)})$ assuming that it is symmetrical and that the conditions

$$\Sigma\Delta_1 > 0, \ldots, \Sigma\Delta_{i+3} > 0 \tag{1.8}$$

are satisfied.

By virtue of the assumption of the induction, the positive-definiteness of the quadratic form $\varphi^{(i)}(x^{(i)})$ and the satisfaction of conditions (1.5), which take the form

$$a_1^{(i)} > 0, \ \ldots, \ a_i^{(i)} > 0 \tag{1.9}$$

follow from the first i conditions of (1.8).

Consider the quadratic form

$$\Phi(\rho) = \varphi^{(i+3)}(x^{(i+3)}) + \rho(x_{i+1}^2 + x_{i+2}^2 + x_{i+3}^2) = x^{(i+3)T} \begin{Vmatrix} A^{(i)} & B \\ B^T & D + \rho E_3 \end{Vmatrix} x^{(i+3)}$$

where the matrices B and D contain elements which extend the matrix $A^{(i)}$ to $A^{(i+3)}$.

In the case of this matrix, relation (1.4) has the form

$$\begin{vmatrix} A^{(i)} + \sigma E_i & B \\ B^T & D + (\rho + \sigma)E_3 \end{vmatrix} = \sigma^{i+3} + (3\rho + a_1^{(i+3)})\sigma^{i+2} + (3\rho^2 + \rho\Sigma a_1^{(i+2)} + a_2^{(i+3)})\sigma^{i+1} +$$
$$+ (\rho^3 + \rho^2 \Sigma a_1^{(i+1)} + \rho\Sigma a_2^{(i+2)} + a_3^{(i+3)})\sigma^i + (\rho^3 a_1^{(i)} + \rho^2 \Sigma a_2^{(i+1)} + \rho\Sigma a_3^{(i+2)} + a_4^{(i+3)})\sigma^{i-1} + \ldots$$
$$\ldots + \rho^3 a_i^{(i)} + \rho^2 \Sigma a_{i+1}^{(i+1)} + \rho \Sigma a_{i+2}^{(i+2)} + a_{i+3}^{(i+3)} \tag{1.10}$$

In the case of fairly large positive ρ, the coefficients of the different powers of σ in (1.10) will be positive, since they are polynomials in powers of ρ with positive coefficients (by virtue of conditions (1.9)) of the leading powers of ρ. Consequently, on the basis of criterion (1.5), the quadratic form $\Phi(\rho)$

will be positive definite with respect to $x^{(i+3)}$. In order that it should remain positive definite when $\rho = 0$, it is sufficient that the roots (with respect to ρ) of the determinant of its matrix

$$f(\rho) = \begin{vmatrix} A^{(i)} & B \\ B^T & D+\rho E_3 \end{vmatrix} = \rho^3 a_i^{(i)} + \rho^2 \Sigma a_{i+1}^{(i+1)} + \rho \Sigma a_{i+2}^{(i+2)} + a_{i+3}^{(i+3)} \tag{1.11}$$

should be negative or, what is the same thing, that the coefficients of the polynomial (1.11) should be positive. Taking account of the relation

$$a_{i+1}^{(i+1)} = \Delta_{i+1}, \quad a_{i+2}^{(i+2)} = \Delta_{i+2}, \quad a_{i+3}^{(i+3)} = \Delta_{i+3}$$

we conclude that the last three coefficients in equality (1.11) are exactly equal to the symmetrized left-hand sides of the last three inequalities of the Silvester criterion (the first is positive by virtue of conditions (1.9)), which proves the positive-definiteness of the quadratic form $\varphi^{(i+3)}(x^{(i+3)})$.

Remark 1. In practice, it is sometimes convenient to calculate the quantities Δ_{i+3}, $\Sigma\Delta_{i+2}$, $\Sigma\Delta_{i+1}$ by differentiating the determinant (1.11) with respect to ρ

$$\Delta_{i+3} = f(0), \quad \Sigma\Delta_{i+2} = df(0)/d\rho, \quad \Sigma\Delta_{i+1} = \tfrac{1}{2} d^2 f(0)/d\rho^2$$

Remark 2. The positive-definiteness of the function $\varphi^{(i+1)}(x^{(i+1)})$ could be proved in the same way as for $\varphi^{(i+3)}(x^{(i+3)})$ and not on the basis of the Silvester criterion. The necessity can also be proved without using the Silvester criterion by the same induction and taking account of the relation $a_i^{(i)} = \Delta_i$. In particular, the proof of the Silvester criterion will then follow from this proof.

Remark 3. The proof in the case of symmetry with $s = 2$ is obvious by simplifying the proof for $s = 3$ carried out above.

2. SYMMETRIZATION OF THE MANN CRITERION FOR THE POSITIVE-DEFINITENESS OF A QUADRATIC FORM IN A LINEAR MANIFOLD

We will now consider the problem of the positive-definiteness of a quadratic form in a linear manifold

$$\varphi(x) = x^T A x \quad (A^T = A), \quad u = Bx = 0$$

where x and u are n-dimensional vectors, $k < n$, and we introduce the quadratic form

$$\psi = \varphi + \rho u^2$$

where ρ is a positive number as large as desired, representing it in the two different forms

$$\psi = f(x) = x^T A x + \rho x^T B^T B x = x^T F x$$

and

$$\psi = g(u, x) = x^T A x + \rho(2u^T B x - u^2) = y^T G y$$

where

$$F = A + \rho B^T B, \quad G = \begin{Vmatrix} -\rho E_k & \rho B \\ \rho B^T & A \end{Vmatrix}, \quad y = \begin{Vmatrix} u \\ x \end{Vmatrix}$$

We will calculate the determinant of the matrix G, for which we add its first row, multiplied by B^T on the left, to the second row and expand it with respect to the first column. We obtain

$$\det G = \begin{vmatrix} -\rho E_k & \rho B \\ 0 & F \end{vmatrix} = (-\rho)^k \det F \tag{2.1}$$

On the other hand

$$\det G = \rho^{2k} \det H(\varepsilon), \quad H(\varepsilon) = \begin{Vmatrix} -\varepsilon E_k & B \\ B^T & A \end{Vmatrix} = H(0) + o(1) \tag{2.2}$$

$$\varepsilon = \frac{1}{\rho}, \quad H(0) = D = \begin{Vmatrix} 0 & B \\ B^T & A \end{Vmatrix}$$

Comparing relations (2.1) and (2.2), we obtain

$$\det F = (-1)^k \det D \rho^k + \dots$$

Terms with powers of ρ less than k are indicated by dots.

By exactly the same arguments with the truncated quadratic form $\psi^{(i)}(x^{(i)})$, obtained by successively equating the last $n-i$ components of the vector x to zero, it can be shown that similar relations hold for the principal diagonal minors δ_i and Δ_i of order i of the matrices F and D

$$\delta_i = (-1)^k \Delta_{k+i} \rho^k + \dots, \quad i = n, \dots, k+1 \tag{2.3}$$

$$\delta_i = \delta_i^\circ \rho^i + \dots, \quad i = k, \dots, 1 \tag{2.4}$$

Here, δ_i° is the sum of the squares of all possible minors of order i of the first i columns of the matrix B.

The symmetrized Silvester criterion (1.3) for the positive-definiteness of the quadratic form $f(x)$ takes the form

$$\Sigma \delta_i > 0, \quad i = 1, \dots, n \tag{2.5}$$

By virtue of relations (2.4), the first k conditions of (2.5) are satisfied in the case of non-degeneracy of the determinant composed of the first k columns of matrix B. This constraint can be replaced by the requirement of the completeness of the rank of the matrix which comprises the first $k+1$ columns of the matrix B, since the following principal diagonal minors of order $i > k$ do not change under a permutation of the first $k+1$ rows in the matrix F with the same simultaneous permutation of its first $k+1$ columns. If, in the quadratic form $\varphi(x)$, the variables $x_{k+1}, x_{k+2}, x_{k+3}$ constitute a cyclic group, the first k inequalities in conditions (2.5) will be satisfied, subject to the requirement of the completeness of the rank of the matrix which is made up of the first $k+3$ columns of the matrix B.

On comparing the last $n-k$ inequalities in (2.5) with relations (2.3) and taking account of the identity $\psi(x) = \varphi(x)$ when $u = Bx = 0$, we obtain the symmetrized Mann criterion for the positive-definiteness of the quadratic form $\varphi(x)$ in the linear manifold $u = Bx = 0$.

$$(-1)^k \Sigma \Delta_i > 0, \quad i = 2k+1, \dots, n+k \tag{2.6}$$

which is the main result of this section.

If the Silvester criterion (1.2) is taken as the criterion for the positive-definiteness of the quadratic form $f(x)$ instead of conditions (1.3), the same arguments provide a proof of the Mann criterion [2]

$$(-1)^k \Delta_i > 0, \quad i = 2k+1, \dots, n+k \tag{2.7}$$

The Mann criterion has been repeatedly proved (see [3], for example). The derivation which has been presented above provides a further method for proving this criterion. The matrix D, which occurs in the Mann criterion, was examined long ago by Weierstrass in the problem of a conditional minimum [4].

The Weierstrass criterion is formulated in terms of the eigenvalues of a quadratic form $\varphi(x)$ in a linear manifold $u = 0$ and is equivalent to the inequalities

$$a_i > 0, \quad i = 2k+1, \dots, n+k \tag{2.8}$$

where a_i are the coefficients of the equation

$$\Delta(\sigma) = \begin{vmatrix} 0 & B \\ B^T & A + \sigma E_n \end{vmatrix} = (-1)^k (a_{2k}\sigma^{n-k} + a_{2k+1}\sigma^{n-k-1} + \dots + a_{n+k}) = 0 \tag{2.9}$$

The coefficients a_i represent the sums of all possible diagonal minors of the ith order with the sign $(-1)^k$ of the matrix D which border its principal diagonal minor of the kth order, consisting of zeros. The coefficient a_{2k} is equal to the sum of the squares of the determinants of order k, comprising the columns of the matrix B. Actually, the Weierstrass criterion corresponds to the complete symmetrization of the Mann criterion, and only requires the completeness of the rank of the matrix B.

We will now give a convenient formula for evaluating the determinant

$$\Delta = \begin{vmatrix} 0 & B \\ B^T & A \end{vmatrix}$$

which is obtained by an expansion in minors of its first k rows and columns

$$(-1)^k \Delta = \Sigma A_{s^i s^j} \tilde{B}_{s^i} \tilde{B}_{s^j}, \quad A_{s^i s^j} = A_{s^j s^i}, \quad \tilde{B}_{s^i} = (-1)^{s_1^i + \ldots + s_k^i} B_{s^i} \tag{2.10}$$

Here, B_{s^i} ($i = 1, \ldots, m$) are all non-zero determinants composed of columns with the numbers $s^i = (s_1^i, s_2^i, \ldots, s_k^i)$ of the matrix B, and $A_{s^i s^j}$ are the minors of the matrix A, which are obtained by deleting from it the rows with the numbers $s^i = (s_1^i, s_2^i, \ldots, s_k^i)$ and the columns with the numbers $s^j = (s_1^j, s_2^j, \ldots, s_k^j)$.

Remark 4. The obvious relations

$$a_{2k+i} = (-1)^k \frac{d^{n-k-i} \Delta(0)}{(n-k-i)! d\sigma^{n-k-i}}$$

may also be useful when calculating the coefficients a_i from inequalities (2.8).

3. EXAMPLE

As an example, we will consider the problem of the stability of the relative equilibrium orientations of a satellite gyrostat with a subsatellite suspended from it on a long massless tether in an orbital system of coordinates for a circular orbit. This problem is a generalization of the problem considered earlier in [5]. By the term gyrostat, we mean a system consisting of a rigid body and symmetrical rotors (gyrodynes) with fixed axes of rotation similar to the "Mir" space station. For simplicity, we will assume that the motion of the system relative to its centre of mass does not affect the orbit of the centre of mass, that the tether length is much greater than the dimensions of the satellite and much less than the radius of the orbit of the centre of mass, and that the mass of the subsatellite is much less than the mass of the satellite.

We will determine the equilibrium orientations and the conditions for their stability from the conditions for a minimum of the changed potential energy of the gravitational and inertial forces acting on the satellite which, under the assumptions which have been made, can be written in the form

$$W(\gamma, \beta) = \Sigma \left\{ \frac{3}{2} \omega^2 A_1 \gamma_1^2 - \frac{1}{2} \omega^2 A_1 \beta_1^2 - 3m\omega^2 L l_1 \gamma_1 - \omega k_1 \beta_1 \right\}$$

Here ω is the Kepler orbital angular velocity, m is the mass of the subsatellite, L is the length of the tether A_i are the moments of inertia of the satellite with the rotors about the principal central axes of inertia x_i; γ_i, β_i are the projections of the radius vector of the centre of mass of the satellite and the vector of the normal to the orbital plane onto the axes of the unit vectors γ and β, and k_i, l_i are the projections of the gyrostatic momentum vector k (the vector of the relative angular momentum of the rotors, which is assumed to be constant) and the radius vector of the point of suspension of the tether relative to the centre of mass of the satellite.

In the expression for W, the equilibrium vertical orientation of the tether has been taken into account. This is not inconsistent with the general problem of a minimum, since this orientation of the tether yields a minimum in the changed potential energy for any fixed values of the direction cosines γ_i, β_i.

The variables γ_i, β_i are connected by the obvious geometrical relations

$$\psi = \gamma^2 - 1 = 0, \quad \chi = \gamma\beta = 0, \quad \varphi = B^2 - 1 = 0 \tag{3.1}$$

Introducing the function V with the Lagrange multipliers λ, μ, ν

$$V(\lambda,\mu,\nu,\gamma,\beta)=\frac{1}{\omega}W+\lambda\chi+\frac{1}{2}\mu\psi+\frac{1}{2}\nu\varphi=$$

$$=\Sigma\left\{\frac{1}{2}(\mu+3A_1)\gamma_1^2+\frac{1}{2}(\nu-A_1)\beta_1^2-q_1\gamma_1-p_1\beta_1\right\}\quad q=3mLl,\quad p=\frac{1}{\omega}k$$

we write the equilibrium equations in the form

$$\partial V/\partial\gamma_i=(\mu+3A_i)\gamma_i+\lambda\beta_i-q_i=0,\quad i=1,2,3 \tag{3.2}$$

$$\partial V/\partial\beta_i=\lambda\gamma_i+(\nu-A_i)\beta_i-p_i=0,\quad i=1,2,3 \tag{3.3}$$

It is clear from Eqs (3.2) and (3.3) that any specified orientation of the satellite, which is determined by the direction cosines (γ_i,β_i), that satisfy conditions (3.1), can be made the equilibrium orientation by the suitable choice of the parameters q_i, p_i and that there is a certain arbitrariness in this choice, which is determined by the free parameters λ, μ, ν.

The parameters μ and ν have a simple geometrical meaning. On multiplying Eqs (3.2) by γ_i and Eqs (3.3) by β_i and summing over $i = 1, 2, 3$, taking account of relations (3.1), we obtain

$$\mu=\Sigma q_1\gamma_1-3\Sigma A_1\gamma_1^2,\quad \nu=\Sigma p_1\beta_1+\Sigma A_1\beta_1^2 \tag{3.4}$$

The geometrical meaning of λ is less obvious.

The equilibrium orientations considered on the basis of the Lagrange and Kelvin theorems [1] will be stable if the specified values of γ_i, β_i turn the function W into a minimum under the conditions (3.1). For this to be so, it is sufficient that the quadratic part of the function V

$$\varphi(x) = \delta^2V/2 = x^TAx$$

should be positive definite in the linear manifold $Bx = 0$, where

$$x=\begin{Vmatrix}\delta\gamma\\ \delta\beta\end{Vmatrix},\quad A=\frac{\partial^2V}{\partial(\gamma,\beta)^2}=\begin{Vmatrix}D & \lambda E_3\\ \lambda E_3 & C\end{Vmatrix}$$

$$B=\frac{\partial(\chi,\psi,\varphi)}{\partial(\gamma,\beta)}=\begin{Vmatrix}\beta_1 & \beta_2 & \beta_3 & \gamma_1 & \gamma_2 & \gamma_3\\ \gamma_1 & \gamma_2 & \gamma_3 & 0 & 0 & 0\\ 0 & 0 & 0 & \beta_1 & \beta_2 & \beta_3\end{Vmatrix} \tag{3.5}$$

$$C=\frac{\partial^2V}{\partial\gamma^2}=\operatorname{diag}(c_1,c_2,c_3),\quad D=\frac{\partial^2V}{\partial\beta^2}=\operatorname{diag}(d_1,d_2,d_3)$$

$$c_i=\nu-A_i,\quad d_i=\mu+3A_i$$

We will use criterion (2.6) to solve the problem. In the case of this criterion, we initially calculate the determinant

$$-\Delta=-\begin{vmatrix}0 & B\\ B^T & A\end{vmatrix}$$

Using formula (2.10), we obtain 18 non-zero determinants which can be made up from the columns of the matrix B

$$\tilde B_{i56}=(-1)^{i+1}\gamma_i\alpha_1,\quad \tilde B_{i46}=(-1)^{i+1}\gamma_i\alpha_2,\quad \tilde B_{i45}=(-1)^{i+1}\gamma_i\alpha_3 \tag{3.6}$$

$$\tilde B_{23i+3}=(-1)^i\beta_i\alpha_1,\quad \tilde B_{13i+3}=(-1)^i\beta_i a_2,\quad \tilde B_{12i+3}=(-1)^i\beta_i\alpha_3,\quad i=1,2,3$$

The following non-zero additional minors of the matrix A ($A_{s_is_j} = A_{s_js_i}$) correspond to these determinants

$$
\begin{aligned}
&A_{156\ 156} = c_1d_2d_3, \quad A_{256\ 256} = c_1d_1d_3 - d_3\lambda^2, \quad A_{356\ 356} = c_1d_1d_2 - d_2\lambda^2 \\
&A_{234\ 234} = d_1c_2c_3, \quad A_{235\ 235} = d_1c_1c_3 - c_3\lambda^2, \quad A_{236\ 236} = d_1c_1c_2 - c_2\lambda^2 \\
&A_{156\ 234} = \lambda^3, \quad A_{235\ 256} = A_{236\ 356} = \lambda^3 - c_1d_1\lambda \\
&A_{246\ 345} = (-1)^1 d_1\lambda^2, \quad A_{135\ 126} = (-1)^1 c_1\lambda^2 \\
&A_{246\ 126} = (-1)^1 c_2d_3\lambda, \quad A_{345\ 135} = (-1)^1 c_3d_2\lambda
\end{aligned}
\tag{3.7}
$$

and, also, the minors which are obtained from formulae (3.7) by simultaneous cyclic permutation of the indices (123) and (456). Here, the exponent accompanying the minus one is also to be considered as a cyclically varying index.

Substituting the values of (3.6) and (3.7) into the equality (2.10) we obtain

$$
\begin{aligned}
&-\Delta = \Sigma d_1\alpha_1^2 \Sigma\beta_1^2 c_2c_3 + \Sigma c_1\alpha_1^2 \Sigma\gamma_1^2 d_2d_3 - \lambda^2(\Sigma c_1\gamma_1^2 + \Sigma d_1\beta_1^2) \\
&\alpha_1 = \beta_2\gamma_3 - \beta_3\gamma_2
\end{aligned}
\tag{3.8}
$$

Taking account of expressions (3.5) for c_i, we represent expression (3.8) in the form of a quadratic trinomial in powers of ν

$$
\begin{aligned}
&-\Delta(\nu) = a\nu^2 + b\nu + c \\
&a = \Sigma d_1\alpha_1^2, \quad b = -a\Sigma(A_2 + A_3)\beta_1^2 + \Sigma d_2d_3\gamma_1^2 - \lambda^2 \\
&c = a\Sigma A_2A_3\beta_1^2 - \Sigma A_1\alpha_1^2 \Sigma d_2d_3\gamma_1^2 - \lambda^2(\Sigma d_1\beta_1^2 - \Sigma A_1\gamma_1^2)
\end{aligned}
$$

On writing out determinant (1.4) for this case

$$
a(\nu + \sigma)^2 + b(\nu + \sigma) + c
$$

and taking account of Remark 1 on the differentiation of determinant (1.11), we can write the stability conditions in the form

$$
a > 0, \quad 2a\nu + b > 0, \quad a\nu^2 + b\nu + c > 0 \tag{3.9}
$$

Determinant (3.8) can also be represented in the form of a quadratic trinomial in powers of μ

$$
\begin{aligned}
&-\tilde{\Delta}(\mu) = \tilde{a}\mu^2 + \tilde{b}\mu + \tilde{c} \\
&\tilde{a} = \Sigma c_1\alpha_1^2, \quad \tilde{b} = 3a\Sigma(A_2 + A_3)\gamma_1^2 + \Sigma c_2c_3\beta_1^2 - \lambda^2 \\
&\tilde{c} = 9a\Sigma A_2A_3\gamma_1^2 + 3\Sigma A_1\alpha_1^2 \Sigma c_2c_3\beta_1^2 - \lambda^2(\Sigma c_1\gamma_1^2 + 3\Sigma A_1\beta_1^2)
\end{aligned}
$$

The stability conditions then take the form

$$
\tilde{a} > 0, \quad 2\tilde{a}\mu + \tilde{b} > 0, \quad \tilde{a}\mu^2 + \tilde{b}\mu + \tilde{c} > 0 \tag{3.10}
$$

The calculations presented demonstrate the use of the criteria of positive-definiteness of symmetrical quadratic forms, which have been obtained in the preceding sections. Here, we shall not concern ourselves with analysing the stability conditions which have been obtained and with determining the optimal values of the parameters p and q which ensure equilibrium and stability. We shall merely confine ourselves to the remark that conditions (3.9) and (3.10) can be satisfied for any required orientation of the satellite owing to the choice of the free parameters μ, ν, and λ.

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